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On the hydromagnetic stability of an unsteady Kelvin–Helmholtz flow

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An analysis is made of the stability of an unsteady basic flow of a conducting fluid in the presence of a parallel magnetic field. The particular profile investigated is the classical Kelvin–Helmholtz profile modified by the addition of an oscillatory component. Two cases are considered in detail: that of a perfectly conducting fluid and that of a poorly conducting fluid. The investigation leads, in both cases, to an equation of the Hill type. It is concluded that the magnetic field has a stabilizing influence but is nevertheless unable to suppress the Kelvin–Helmholtz instability in an unsteady (basic) flow.

1. Introduction

The subject of the stability of unsteady flows is of some current interest. However, the question of what is meant by the stability (or instability) of a time-dependent basic flow is itself unsettled, though some discussion of this matter may be found in Conrad & Criminale (1965) and in Shen (1961). This degree of uncertainty in the definition of stability does not exist in the case of a flow which has an oscillatory time dependence. It is with such a flow that we are concerned here. For a basic flow consisting of a steady component together with a component that is periodic in time the flow is stable if all (small) perturbations remain small for all time, unstable if they do not.

The type of behaviour that the oscillatory component in the basic flow gives rise to is conveniently illustrated by the problem of a simple pendulum which is constrained to move under the influence of a vertical oscillation applied to the support end of the pendulum (see Stoker 1950). Such a system is described by an equation of the Mathieu type. In the case of a fluid flow a somewhat related situation has been investigated by Kelly (1965). Kelly considered the effect of an oscillatory component in the basic velocity on the stability of the classical Kelvin–Helmholtz profile. Again, the Mathieu equation arises.

The use of a discontinuous profile (such as in the work of Kelly) in fluid stability theory warrants some comment. In the analysis of such a complex situation as the stability of an unsteady flow any simplification is welcome, provided no undue distortion arises. Drazin (1961), in an investigation of the use of discontinuous profiles in relation to the Orr–Sommerfeld equation, has shown that such a practice is physically justifiable for long waves. Thus, in treating the stability of the (unsteady) Kelvin–Helmholtz profile we shall adopt this outlook and our results will only be physically realistic for long waves.

The effect of a magnetic field on the stability of a *steady* flow field has received considerable attention (see, for example, Chandrasekhar 1961). For an *unsteady* flow field no such study exists. The problem is of interest for its novelty: effects arise in the analysis of time-dependent basic flows that are entirely absent in their steady counterparts. Also, coupled with the effect of the oscillatory component in the basic velocity is the effect of the magnetic field as it interacts with the flow. We shall consider such interactions in this paper.

The effect of an *oscillatory* magnetic field (in addition to a steady component) on a *steady* velocity distribution has been investigated by Drazin (1967) with the intention of determining whether such an oscillatory field is more stabilizing than the corresponding steady one. He concludes that, for a non-dissipative vortex sheet, it is 'less efficient to stabilize the vortex sheet with an oscillatory field than a steady one'. We shall take this to be the case in our treatment, which includes magnetic dissipation, and therefore consider only a *steady* basic magnetic field. A more complete treatment would, however, be of some interest.

In this paper we investigate the stability of an *unsteady* basic flow of a conducting fluid in the presence of a *steady* magnetic field. Following Kelly (1965), we consider the basic flow to be the time-dependent Kelvin–Helmholtz one, and the magnetic field is taken to be parallel to the basic flow. To investigate the effect of the field we consider in detail the two extremes of a large and small magnetic Reynolds number. The fluid has a density stratification, and both two- and three-dimensional disturbances are considered. The case of a basic flow with an oscillatory component small compared with the mean flow is considered in detail.

2. Basic equations

We consider an incompressible inviscid fluid of density ρ, uniform conductivity σ and magnetic permeability μ. In a Cartesian co-ordinate system (x_1, x_2, x_3) the basic state that we wish to consider is

$$\left.\begin{aligned} \mathbf{v} &= \mathbf{v}_0 = (v_0(x_2, \tau), 0, 0), \\ \mathbf{H} &= \mathbf{H}_0 = (H_0, 0, 0), \quad H_0 \text{ a constant,} \end{aligned}\right\} \tag{2.1}$$

where $\mathbf{v}$ denotes the fluid velocity, $\mathbf{H}$ the magnetic field and τ time. The time-dependent shear profile of interest is

$$v_0 = \begin{cases} U_0 + \epsilon_1 \cos\omega_1\tau, & x_2 > 0, \\ -U_0 + (\rho_1\epsilon_1/\rho_2)\cos\omega_1\tau, & x_2 < 0, \end{cases} \tag{2.2}$$

where the density in the basic state is given by

$$\rho = \begin{cases} \rho_1, & x_2 > 0, \\ \rho_2, & x_2 < 0. \end{cases} \tag{2.3}$$

The quantities ρ_1, ρ_2, ω_1, ϵ_1 and U_0 are all constants. The amplitudes occurring in (2.2) are chosen so that the pressure gradient in the basic state is continuous across the interface $x_2 = 0$.

To describe perturbations of this basic state we put

$$\mathbf{v}_1 = (v_1(x_2,\tau),\, v_2(x_2,\tau),\, v_3(x_2,\tau)) \exp(ik_1x_1+ik_3x_3),$$
$$\mathbf{h}_1 = (h_1(x_2,\tau),\, h_2(x_2,\tau),\, h_3(x_2,\tau)) \exp(ik_1x_1+ik_3x_3),$$

where $\mathbf{v}_1$ denotes the perturbation velocity and $\mathbf{h}_1$ the perturbation magnetic field. Here k_1 and k_3 are positive constants. Then it may be shown (see Drazin 1967; Stuart 1954) that v and ψ satisfy the (dimensionless) equations

$$L\psi - i\alpha v = R_m^{-1}\Delta\psi, \tag{2.4}$$

$$L\Delta v - i\alpha(\partial^2 U/\partial y^2)\, v = i\alpha A^2 \Delta\psi, \tag{2.5}$$

where

$$v_2 = U_0 v, \quad v_0 = U_0\, U(y,t), \quad h_2 = H_0\psi, \quad x_1 = lx, \quad x_2 = ly, \quad x_3 = lz,$$
$$\alpha = lk_1, \quad \gamma = lk_3, \quad \tau = lU_0^{-1}t, \quad R_m = 4\pi\mu\sigma l U_0, \quad A^2 = \mu H_0^2/(4\pi\rho U_0^2),$$

and l is a characteristic length scale. The operators L and Δ are defined by

$$L \equiv (\partial/\partial t) + i\alpha U, \quad \Delta \equiv (\partial^2/\partial y^2) - \lambda^2,$$

where $\lambda^2 = \alpha^2+\gamma^2$. In the above R_m is the magnetic Reynolds number and A the Alfvén number.

The velocity profile of interest, equation (2.2), may now be expressed in the form

$$U = \begin{cases} U_1(t) = 1+\epsilon\rho_2\cos\omega t, & y > 0, \\ U_2(t) = -1+\epsilon\rho_1\cos\omega t, & y < 0, \end{cases} \tag{2.6}$$

where ϵ and ω are positive constants.

To investigate the stability of the basic state (2.6) it is necessary to consider the nature of the solution of (2.4) and (2.5) subject to the appropriate boundary conditions. These conditions may be obtained as follows.

In the basic flow the interface is the plane $y = 0$; in the disturbed flow we denote the interface by $y = \xi_1(x,z,t)$. Kinematically, we require (see Lamb 1932, p. 7) that

$$\left(\frac{\partial}{\partial t}+\mathbf{v}.\nabla\right)(y-\xi_1) = 0,$$

which, when linearized, gives

$$L\xi = v(0,t) \quad \text{at} \quad y = 0, \tag{2.7}$$

where $\xi_1 = \xi(t)\exp(i\alpha x+i\gamma z)$. Equation (2.7) expresses the continuity of the normal component of the velocity across the interface.

In addition to (2.7) we require two further boundary conditions. Clearly, one such condition is provided by the requirement of boundedness of v and ψ:

$$v, \psi \to 0 \quad \text{as} \quad |y| \to \infty. \tag{2.8}$$

The remaining boundary condition may be obtained by considering the first integral of (2.5) and use of (2.4). In §§ 3 and 4 we shall consider two special cases of (2.4) and derive the remaining boundary condition as appropriate.

3. Ideally conducting fluid

If the fluid may be considered as a perfect conductor then we may approximate (2.4) by

$$i\alpha v = L\psi, \tag{3.1}$$

and (2.5) may be expressed solely in terms of ψ:

$$L\Delta L\psi + \alpha^2 A^2 \Delta\psi - i\alpha(\partial^2 U/\partial y^2)\, L\psi = 0. \tag{3.2}$$

For the velocity profile (2.6) equation (3.2) reduces to

$$(L^2 + \alpha^2 A^2)\,\Delta\psi = 0, \tag{3.3}$$

which has a solution, bounded at infinity, of the form

$$\psi(y,t) = \begin{cases} f(t)\, e^{-\lambda y} & (y > 0), \\ g(t)\, e^{\lambda y} & (y < 0), \end{cases} \tag{3.4}$$

where f and g are to be determined by the conditions pertaining at the interface. These conditions consist of that given by (2.7), and the condition

$$\left[\rho(L^2 + \alpha^2 A^2)\,\frac{\partial\psi}{\partial y}\right] = 0, \tag{3.5}$$

where the notation

$$[\phi(y,t)] \equiv \phi(0+,t) - \phi(0-,t)$$

for any function ϕ is used. Boundary condition (3.5) follows from consideration of the first integral of (3.2) across the interface. Since for both $y > 0$ and $y < 0$ the term $\partial^2 U/\partial y^2$ is zero, the only contribution this term makes is through the boundary conditions. This method of determining the appropriate boundary condition is similar to that used by Drazin (1961).

Application of conditions (2.7) and (3.5) to the form (3.4) leads to an equation for $\xi(t)$, namely

$$\frac{d^2\xi}{dt^2} + 2i\alpha(\alpha_1 U_1 + \alpha_2 U_2)\,\frac{d\xi}{dt} + i\alpha\left(\alpha_1\frac{dU_1}{dt} + \alpha_2\frac{dU_2}{dt}\right)\xi + \alpha^2(\bar{A}^2 - \alpha_1 U_1^2 - \alpha_2 U_2^2)\,\xi = 0, \tag{3.6}$$

where

$$\alpha_1 = \rho_1/(\rho_1+\rho_2), \quad \alpha_2 = \rho_2/(\rho_1+\rho_2), \quad \bar{A}^2 = \mu H_0^2/[2\pi U_0^2(\rho_1+\rho_2)].$$

The substitution

$$\xi(t) = \hat{\xi}(t)\exp\left(-i\alpha\int_0^t (\alpha_1 U_1 + \alpha_2 U_2)\,dt\right)$$

reduces (3.6) to the more convenient form

$$(d^2\hat{\xi}/dt^2) + \alpha^2(\bar{A}^2 - \alpha_1\alpha_2(U_1 - U_2)^2)\,\hat{\xi} = 0. \tag{3.7}$$

The case of a time-dependent shear profile for the flow *in the absence of a magnetic field* has been investigated by Kelly (1965). Kelly allowed for the effects of gravity and surface tension and derived an equation similar to (3.7). Comparison of this equation (Kelly's equation (3.11)) with (3.7) shows that the effect of a magnetic field is equivalent to a surface tension T ($T \equiv \mu k_1^2 H_0^2/[2\pi(k_1^2 + k_3^2)^{\frac{3}{2}}]$), a result well known in this connexion. Kelly's discussion of the effect of surface tension on the unsteady Kelvin–Helmholtz flow is therefore also applicable (with slight changes) to the hydromagnetic system considered here. In order to facilitate comparison with Kelly's results it is convenient to include the effects of gravity $\mathbf{g} = (0, -g, 0)$ and surface tension T in our analysis.†

For the velocity profile (2.4), with $\epsilon \neq 0$, equation (3.7) is of the Hill type (see Whittaker & Watson 1969, p. 406). In terms of the variable $\hat{t} = \frac{1}{2}\omega t$, *and allowing for the effects of gravity and surface tension*, the modified version of (3.7) may be written in the form

$$(d^2\xi/d\hat{t}^2) + (\theta_0 + 2\theta_1 \cos 2\hat{t} + 2\theta_2 \cos 4\hat{t})\,\xi = 0, \tag{3.8}$$

where

$$\theta_0 = (4/\omega^2)\,[\lambda(\alpha_2-\alpha_1)\,F^{-1} + \alpha^2\bar{A}^2 + \lambda^3\alpha_1\, W^{-1} - 4\alpha^2\alpha_1\alpha_2 - \tfrac{1}{2}\alpha^2\alpha_1\alpha_2\epsilon^2(\rho_2-\rho_1)^2],$$

$$\theta_1 = -8\alpha_1\alpha_2(\rho_2-\rho_1)\,\epsilon(\alpha/\omega)^2, \quad \theta_2 = -\alpha_1\alpha_2(\rho_2-\rho_1)^2\epsilon^2(\alpha/\omega)^2$$

and

$$F^{-1} = lU_0^{-2}g, \quad W^{-1} = l^{-1}U_0^{-2}\rho^{-1}T.$$

When the velocities U_1 and U_2 are constant (i.e. when $\epsilon = 0$) (3.8) shows that the interface is stable if

$$\lambda(\alpha_2-\alpha_1)\,F^{-1} + \alpha^2\bar{A}^2 + \lambda^3\alpha_1 W^{-1} > \alpha^2\alpha_1\alpha_2(U_1-U_2)^2. \tag{3.9}$$

If this expression is minimized with respect to the wavenumber α, then the instability is suppressed for

$$\bar{A}^2 + 2\{\alpha_1(\alpha_2-\alpha_1)\,F^{-1}W^{-1}\}^{\frac{1}{2}} > \alpha_1\alpha_2(U_1-U_2)^2. \tag{3.10}$$

In the absence of surface tension this reduces to the standard (hydromagnetic) Kelvin–Helmholtz stability criterion (Chandrasekhar 1961, p. 511).

Returning to the time-dependent problem, as described by (3.8), the case of most interest is that arising when the unsteady component of the basic flow is small compared with the steady component. Thus, in the special case

$$|\rho_2-\rho_1|\,\epsilon \ll 1,$$

for which the difference velocity $U_1(t) - U_2(t)$ does not depart greatly from its mean, (3.8) may be approximated by the Mathieu equation

$$(d^2\xi/d\hat{t}^2) + (\theta_0 + 2\theta_1 \cos 2\hat{t})\,\xi = 0, \tag{3.11}$$

where now

$$\theta_0 = (4/\omega)^2\,[\lambda(\alpha_2-\alpha_1)\,F^{-1} + \alpha^2\bar{A}^2 + \lambda^3\alpha_1 W^{-1} - 4\alpha^2\alpha_1\alpha_2],$$

$$\theta_1 = -8\alpha_1\alpha_2(\rho_2-\rho_1)\,\epsilon(\alpha/\omega)^2.$$

Mathieu's equation has received considerable attention in the literature (see, for example, McLachlan 1947). In particular, the stability regions of the Mathieu equation are known and, for convenience, are shown in figure 1. It may be seen from figure 1 that instabilities are possible whenever $\theta_0 = n^2$, for integer n. However, in a real fluid, where dissipative effects such as viscosity and conductivity are operative, only the subharmonic response is likely to be significant.

† Note the difference in notation: in Kelly the index 1 refers to the lower fluid and 2 to the upper fluid.

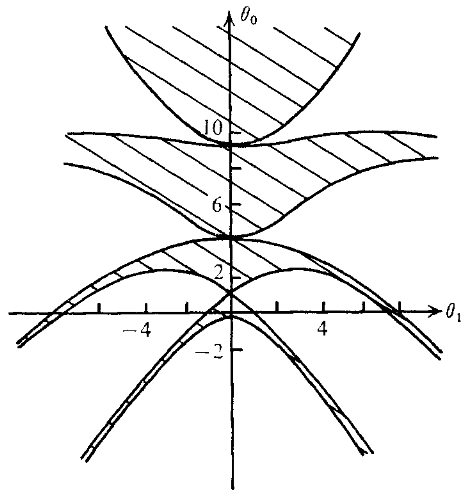

FIGURE 1. Stability diagram for Mathieu's equation. The stable regions are shown hatched.

We shall therefore only consider the stability region (of figure 1) that is bounded by $\theta_0 \doteqdot 1 \pm \theta_1$. Thus, for small θ_1 the subharmonic instability is given by $\theta_0 = 1$, i.e. by

$$\tfrac{1}{4}\omega^2 = \lambda(\alpha_2-\alpha_1)F^{-1}+\alpha^2\bar{A}^2+\lambda^3\alpha_1 W^{-1}-4\alpha^2\alpha_1\alpha_2. \qquad (3.12)$$

When $\theta_0 \doteqdot 1$ the unstable solution of (3.11) behaves like $\exp(\frac{1}{2}|\theta_1|\hat{t})$ for small $|\theta_1|$ (see Whittaker & Watson, p. 424). Thus, the growth rate (in terms of the t variable) for the subharmonic response is

$$\tfrac{1}{4}|\theta_1|\omega = 2\alpha_1\alpha_2|\rho_2-\rho_1|\epsilon\alpha^2/\omega, \qquad (3.13)$$

where ω is given by equation (3.12).† Thus the effect of the magnetic field enters purely through (3.12), and therefore to determine this effect it is sufficient to consider the behaviour of ω. However, an inspection of (3.12) shows that no unexpected features arise from inclusion of the magnetic term: it simply causes an increase in ω and a decrease in the growth rate. Clearly, the general features of ω as a function of wavenumber are as indicated in Kelly (see his figure 2 and equation (3.20)), and so no further investigation is warranted.

4. Poorly conducting fluid

When $R_m \ll 1$ equation (2.4) may be approximated by (Stuart 1954; Hains 1965)

$$i\alpha v + R_m^{-1}\Delta\psi = 0, \qquad (4.1)$$

which when combined with (2.5) gives

$$L\Delta v - i\alpha(\partial^2 U/\partial y^2)v - \alpha^2 A^2 R_m v = 0. \qquad (4.2)$$

Note that the Alfvén number and the magnetic Reynolds number now enter the problem in the form of a single 'interaction' parameter $N = A^2R_m$.

† Equation (3.13) is the corrected (non-dimensional) form of (3.22) in Kelly (1965), which has a slight error: $\omega^{-\frac{1}{2}}$ should read ω^{-1}.

The boundary conditions appropriate to (4.2), for the profile (2.6), are readily found (in a manner similar to that used in the previous section):

$$(a)\quad v \to 0 \quad \text{as} \quad |y| \to \infty;$$
$$(b)\quad v(0,t) = \begin{cases} (d\xi/dt) + i\alpha U_1 \xi, & y = 0+, \\ (d\xi/dt) + i\alpha U_2 \xi, & y = 0-; \end{cases}$$
$$(c)\quad [\rho L(\partial v/\partial y)] = 0.$$

Our problem is to solve (4.2), with $\partial^2 U/\partial y^2 = 0$, in the regions $y > 0$, $y < 0$, using conditions (a)–(c) to match the solution across the interface.

Two alternative expansion procedures will be used, each giving a different range of validity. A complete solution, valid for an unrestricted range of parameters, of the poorly conducting fluid problem is desirable but owing to the complexity of the problem approximate techniques seem inevitable.

4.1. *Expansion in N*

We consider a formal expansion of the type

$$v(y,t) = \sum_{j=0}^{\infty} N^j v^{(j)}(y,t), \quad \xi(t) = \sum_{j=0}^{\infty} N^j \xi^{(j)}(t). \tag{4.3}$$

If these expressions are substituted into (4.2) and also into the boundary conditions (a)–(c), then by equating corresponding powers of N we may obtain an infinite set of equations for the infinite number of unknowns $v^{(0)}, v^{(1)}, v^{(2)}, \ldots$. Equating the zero-order terms (which corresponds to setting $N = 0$) gives, after suitable re-arrangement of terms, an equation for $\xi^{(0)}$. In a similar fashion we may obtain equations for $\xi^{(1)}, \xi^{(2)}, \ldots$. We shall assume that $N \ll 1$ and that an investigation of only the first-order equations (i.e. neglecting terms of order N^2) is sufficient to determine the stability nature of the interface.

The procedure outlined above leads (after some algebra) to the equation

$$\frac{d^2\xi}{dt^2} + \left\{2i\alpha(\alpha_1 U_1 + \alpha_2 U_2) + \frac{\alpha^2}{2\lambda^2}\bar{N}\right\}\frac{d\xi}{dt} + \left\{i\alpha\left(\alpha_1\frac{dU_1}{dt} + \alpha_2\frac{dU_2}{dt}\right) - \alpha^2(\alpha_1 U_1^2 + \alpha_2 U_2^2) + \frac{i\alpha^3}{4\lambda^2}(U_1 + U_2)\bar{N}\right\}\xi = 0, \tag{4.4}$$

where $\bar{N} = \bar{A}^2 R_m$.

The substitution

$$\xi(t) = \hat{\xi}(t)\exp\left\{-\int_0^t \left(i\alpha(\alpha_1 U_1 + \alpha_2 U_2) + \frac{\alpha^2}{4\lambda^2}\bar{N}\right)dt\right\} \tag{4.5}$$

reduces (4.4) to an equation of the Hill type:

$$(d^2\hat{\xi}/d\hat{\tau}^2) + (q_0 + 2q_1\cos 2\hat{\tau} + 2q_2\cos 4\hat{\tau})\hat{\xi} = 0, \tag{4.6}$$

where $\hat{\tau} = \frac{1}{2}\omega t$ and

$$q_0 = \frac{2\alpha^2}{\lambda^2\omega^2(\rho_1+\rho_2)^2}\left\{i\alpha\bar{N}(\rho_2^2 - \rho_1^2) - 8\lambda^2\rho_1\rho_2 - \frac{\alpha^2}{8\lambda^2}\bar{N}^2(\rho_2+\rho_1)^2 - \lambda^2\rho_1\rho_2\epsilon^2(\rho_2-\rho_1)^2\right\},$$

$$q_1 = \frac{\alpha^2(\rho_2-\rho_1)\epsilon}{\lambda^2\omega^2(\rho_2+\rho_1)^2}\{\tfrac{1}{2}i\alpha(\rho_2^2-\rho_1^2)\bar{N} - 8\lambda^2\rho_1\rho_2\},$$

$$q_2 = -\alpha^2\rho_1\rho_2\epsilon^2(\rho_2-\rho_1)^2/\{\omega^2(\rho_2+\rho_1)^2\}.$$

Note that q_0 and q_1 are complex and $q_2 \leqslant 0$.

The solution of (4.4) may be readily described in two special cases. If the fluid is *non-conducting* (so that $\bar{N} = 0$), then (4.4) reduces to (3.6) (with $\bar{A}^2 = 0$) and is therefore covered by the discussion in §3. Also, in the absence of stratification (i.e. for $\rho_1 = \rho_2$) (4.6) reduces to

$$\frac{d^2\xi}{d\hat{\tau}^2} - \frac{4\alpha^2}{\omega^2}\left(1 + \frac{\alpha^2 A^4 R_m^2}{16\lambda^4}\right)\xi = 0. \tag{4.7}$$

It is clear from (4.7), considered in conjunction with (4.5), that the magnetic field cannot suppress the instability.

Floquet theory (Whittaker & Watson 1969, p. 412) gives $\phi_1(\hat{\tau})\exp(\nu\hat{\tau})$ and $\phi_2(\tau)\exp(-\nu\hat{\tau})$, where ϕ_1 and ϕ_2 are periodic functions of $\hat{\tau}$, as the forms of the fundamental solutions of (4.6). The characteristic exponent ν depends upon q_0, q_1 and q_2 and is in general complex. We are interested in the growth of solutions of (4.6) and, therefore, in the real part of ν. It is clear that there is always a growing mode if $\mathrm{Re}(\nu)$, the real part of ν, is non-zero. This growth factor $\exp(\frac{1}{2}|\mathrm{Re}(\nu)|\omega t)$ must be compared with the damping term

$$\exp[-(\alpha^2\bar{A}^2 R_m t)/(4\lambda^2)]$$

which arises from the transformation (4.5). Thus, there is always an unstable mode if

$$2\lambda^2\omega|\mathrm{Re}(\nu)| > \alpha^2\bar{N}. \tag{4.8}$$

In order to obtain any insight into the effect of a magnetic field on the time-dependent Kelvin–Helmholtz instability in a poorly conducting fluid it is therefore necessary to determine the real part of the characteristic exponent ν. Hill's method shows that ν is given by the equation

$$\cosh(\pi\nu) = 1 - 2\Delta\sin^2(\tfrac{1}{2}\pi q_0^{\frac{1}{2}}), \tag{4.9}$$

where Δ is an infinite determinant whose elements are functions of q_0, q_1 and q_2 (see Hill 1886; Whittaker & Watson 1969, p. 415). Note that (4.9) implies that $\cosh(\pi\nu)$ is real if q_0, q_1 and q_2 are real. Therefore, if q_0 and q_1 are real (i.e. if $\bar{N}(\rho_2 - \rho_1) = 0$) then either the real part of ν is zero, or the imaginary part of ν is an integer.

Equation (4.9) for the determination of ν is still intractable because of the difficulty in calculating the determinant Δ. Hill (1886) has obtained an approximate expression for Δ which, in our notation, is

$$\Delta \doteqdot 1 + \pi q_1^2 \cot(\tfrac{1}{2}\pi q_0^{\frac{1}{2}})/\{q_0^{\frac{1}{2}}(1 - q_0)\} \quad (q_0 \neq 1). \tag{4.10}$$

The terms neglected are of order q_1^4, q_2^2, $q_1^2 q_2$ and higher. Note that the accuracy of (4.10) is improved for small $\epsilon|\rho_2 - \rho_1|$; if $\rho_2 = \rho_1$ then $\Delta = 1$.

Thus we find that

$$\cosh(\pi\nu) = \cos(\pi q_0^{\frac{1}{2}}) - \pi q_1^2 \sin(\pi q_0^{\frac{1}{2}})/\{q_0^{\frac{1}{2}}(1 - q_0)\} + O\{\epsilon^4(\rho_2 - \rho_1)^4\}. \tag{4.11}$$

If, further, we neglect terms of order $\epsilon^2(\rho_2 - \rho_1)^2$ then (4.11) gives

$$i\nu \doteqdot \frac{\alpha}{\lambda\omega}\left\{\frac{2i\alpha\bar{N}(\rho_2 - \rho_1)}{\rho_1 + \rho_2} - \frac{\alpha^2\bar{N}^2}{4\lambda^2} - \frac{16\lambda^2\rho_1\rho_2}{(\rho_1 + \rho_2)^2}\right\}^{\frac{1}{2}}, \tag{4.12}$$

where, in taking the square root, the root with positive real part is selected.

If (4.12) is substituted into condition (4.8) and the resulting expression simplified it becomes clear that there is always an unstable mode: the magnetic field is unable to eliminate the instability completely. Under the assumptions made in reaching this conclusion this is to be expected. A more accurate representation of the stability/instability regions may be obtained by investigating (4.11) instead of the approximation (4.12). This would clearly entail a more detailed, possibly numerical, discussion. However, in view of the assumptions made in reaching (4.11) such an investigation does not seem worth while.

4.2. *Expansion in* ϵ

The approximate solution presented in §4.1 suffers from the rather severe restriction imposed by the condition $N \ll 1$, though this does cover the case of Alfvén number of order unity. A solution valid for arbitrary N is therefore desirable. However, in view of the obvious difficulty in solving (4.2) subject to conditions (*a*)–(*c*) it is unlikely that an analytical solution can be found without some degree of approximation. One case of interest is that arising when the amplitude of the basic flow oscillation is small compared with the mean flow. In such a case an approximate solution can be found which does not restrict the parameter N.

We expand $v(y,t)$ and $\xi(t)$ in powers of ϵ:

$$v(y,t) = \sum_{j=0}^{\infty} \epsilon^j v_j(y,t), \quad \xi(t) = \sum_{j=0}^{\infty} \epsilon^j \xi_j(t). \tag{4.13}$$

The zeroth-order solution (found by setting $\epsilon = 0$) is readily shown to be of the form

$$v_0(y,t) = \begin{cases} A_1^0 \exp(-i\alpha ct - n_0 y) & (y > 0), \\ A_2^0 \exp(-i\alpha ct + m_0 y) & (y < 0), \end{cases} \tag{4.14}$$

where A_1^0, A_2^0 and c $(\equiv c_r + ic_i)$ are constants, and

$$n_0 = \{\lambda^2 + i\alpha N/(c-1)\}^{\frac{1}{2}}, \quad m_0 = \{\lambda^2 + i\alpha N/\hat{\rho}(c+1)\}^{\frac{1}{2}}, \quad \hat{\rho} = \rho_2/\rho_1.$$

In evaluating square roots the root with positive real part is chosen, thereby satisfying condition (*a*). The (complex) constant c is determined by the remaining boundary conditions, which, after a little algebra, give the dispersion relation

$$\lambda^2\{(c-1)^4 - \hat{\rho}^2(c+1)^4\} + i\alpha N\{(c-1)^3 - \hat{\rho}(c+1)^3\} = 0. \tag{4.15}$$

For the case of equal densities (i.e. for $\hat{\rho} = 1$) (4.15) reduces to the cubic

$$4c^3 + 3iMc^2 + 4c + iM = 0, \quad M = \alpha N/\lambda^2, \tag{4.16}$$

the roots of which are all purely imaginary. This equation, for the case of a two-dimensional disturbance ($\lambda = \alpha$), has been discussed by Drazin (1960). Consideration of (4.16) shows that the flow is unstable for all M, and that the effect of the magnetic field is to make the flow less unstable.

The stability characteristics of the more general case of a three-dimensional disturbance in a stratified fluid are described by (4.15), from which it is clear that c has a non-zero real part. The effect of a non-zero c_r is to introduce an oscillatory term in the time dependence of the (zeroth-order) disturbance, and as such *does not affect the stability nature of the basic flow.* The role of the magnetic field for a three-dimensional disturbance is determined by the parameter M:

$$M = \alpha N/\lambda^2 = (N/\alpha)\cos^2\theta \leqslant N/\alpha \quad (0 \leqslant 0 < \tfrac{1}{2}\pi), \tag{4.17}$$

where θ is the angle the direction of propagation of the disturbance makes with the basic flow. Now for a two-dimensional disturbance the effect of an increase in M is to reduce the growth rate of the instability. So for the three-dimensional case the effect of the field is also reduced in comparison with that for its two-dimensional counterpart: the field is a 'more efficient stabilizer' of two-dimensional disturbances than of three-dimensional ones.

The first-order problem (found by neglecting terms involving ϵ^2) reduces to the solution of the equations

$$\left(\frac{\partial}{\partial t}+i\alpha\right)\left(\frac{\partial^2 v_1}{\partial y^2}-\lambda^2 v_1\right)-\alpha^2 N v_1 = -i\alpha\rho_2\cos\omega t\left(\frac{\partial^2 v_0}{\partial y^2}-\lambda^2 v_0\right) \quad (y>0), \tag{4.18}$$

$$\left(\frac{\partial}{\partial t}-i\alpha\right)\left(\frac{\partial^2 v_1}{\partial y^2}-\lambda^2 v_1\right)-\frac{1}{\rho}\alpha^2 N v_1 = -i\alpha\rho_1\cos\omega t\left(\frac{\partial^2 v_0}{\partial y^2}-\lambda^2 v_0\right) \quad (y<0), \tag{4.19}$$

subject to the conditions

(*A*) $v_1 \to 0$ as $|y| \to \infty$;

(*B*) $v_1(0+,t) = (d\xi_1/dt)+i\alpha\xi_1+\tfrac{1}{2}\rho_2(A_1^0-A_2^0)\,e^{-i\alpha ct}\cos\omega t,$

$v_1(0-,t) = (d\xi^1/dt)-i\alpha\xi_1+\tfrac{1}{2}\rho_1(A_1^0-A_2^0)\,e^{-i\alpha ct}\cos\omega t;$

(*C*)
$$\left[\rho_1\left(\frac{\partial}{\partial t}+i\alpha\right)\frac{\partial v_1}{\partial y}+i\alpha\rho_1\rho_2\cos\omega t\left(\frac{\partial v_0}{\partial y}\right)\right]_{y=0+} = \left[\rho_2\left(\frac{\partial}{\partial t}-i\alpha\right)\frac{\partial v_1}{\partial y}+i\alpha\rho_1\rho_2\cos\omega t\left(\frac{\partial v_0}{\partial y}\right)\right]_{y=0-}.$$

The system defined by equations (4.18) and (4.19) and conditions (*A*)–(*C*) has a solution of the form

$$\left.\begin{aligned} v_1 &= e^{-i\alpha ct}\{f(y)\cos\omega t+g(y)\sin\omega t\},\\ \eta_1 &= e^{-i\alpha ct}(a\cos\omega t+b\sin\omega t),\end{aligned}\right\} \tag{4.20}$$

where a and b are constants. The details of this solution need not be given; it is sufficient to note that the stability (to first order) is determined by the nature of c, which is given by (4.15).

It is of interest to compare the growth rate of the unstable mode as given by the analyses of §§4.1 and 4.2. The region of common validity is defined by the conditions $N \ll 1$, $|\epsilon\rho_2-\epsilon\rho_1| \ll 1$. Equation (4.12) combined with (4.5) shows that the unstable mode grows exponentially with exponent $\alpha c_i t$, c_i being given by

$$c_i \doteqdot \tfrac{1}{2}(4\hat{\rho}^{\frac{1}{2}}-M)/(1+\hat{\rho}). \tag{4.21}$$

Alternatively, this result may be deduced from (4.15), thereby demonstrating the agreement of the procedures of §§4.1 and 4.2 in the common domain. When $\hat{\rho}=1$ equation (4.21) shows that for the case of equal densities

$$c_i \doteqdot 1-\tfrac{1}{4}M,$$

which is in agreement with the complete solution provided by (4.5) and (4.7).

5. Concluding remarks

We have considered the effect of a magnetic field on the stability of a time-dependent velocity profile which has a discontinuity at $y = 0$. The results obtained from use of this profile are therefore only applicable to a realistic physical situation in the limit of small wavenumber. In the discussion of the magnetic effects we considered two limiting cases: that of a perfectly conducting fluid and that of a poorly conducting fluid.

In a perfectly conducting fluid the magnetic lines of force are 'frozen' into the fluid. This tends to give the fluid a degree of rigidity with the result that the field is, in general, a damping influence on the fluid motion. In the *steady* case this damping effect is such that the magnetic field is able to suppress the Kelvin–Helmholtz instability entirely for sufficiently strong fields. In the *unsteady* case the field acts so as to reduce the extent of the instability but, in sharp contrast to the steady case, it is unable totally to suppress the instability, whatever the magnitude of the magnetic field.

In the case of a poor conductor the magnetic field lines are no longer 'frozen in' and may, in fact, slip through the fluid. As a consequence the field is unable to suppress the instability.

The case of a basic flow with an unsteady component that is small compared with the steady component is of particular interest both from a physical standpoint and a theoretical one. Theoretically, such a case is amenable to approximate analysis and readily yields information on the stability of the basic flow. Practically, such a case is of interest as unsteady flows often arise in this manner: a small oscillation (possibly from an external source) affects the basic steady flow. The problem may also be of interest in the theory of turbulence, as is the case in the problem investigated by Greenspan & Benney (1963). For these reasons the case of a small amplitude modulation has been considered in some detail in §§3 and 4. (However, it must be stated that a discussion of large amplitude effects is also of some interest, and therefore a complete solution of (4.2) would be desirable.)

Two factors are involved in the discussion of the stability of the problem considered in this paper: first, the role of the magnetic field and second, its interaction with the applied oscillation in the basic flow. In partial analogy with the pendulum, which may be stabilized or destabilized by an applied oscillation, the role of the oscillatory component in the basic flow is to induce instability into an otherwise potentially stable situation (given a sufficiently large magnetic field). Again, the oscillatory component in the basic flow drives the instability and this supply of energy ensures that the flow is unstable irrespective of the magnetic field.

Finally, the question of the behaviour of the system in the limit of infinite magnetic field is of interest. For a poorly conducting fluid an examination of (4.16) shows that for the case of equal densities $c \sim i\sqrt{3}$, as $M \to \infty$. Thus, in the limit of infinite magnetic field (i.e. as $H_0 \to \infty$) there exists an unstable mode growing like $\exp(\alpha t/\sqrt{3})$ for non-zero wavenumber. In the case of a perfectly conducting fluid there exists an unstable mode (as may be seen from figure 1) in the limit $\theta_0 \to \infty$, provided $\theta_1 \neq 0$. So, in the limit $H_0 \to \infty$ there exists an unstable mode with non-zero wavenumber. Thus, in the limit $H_0 \to \infty$, unstable modes with non-zero wavenumbers exist in both ranges of conductivity considered in this paper.

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